Floer homology and invariants of Legendrian knots

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2014/05/21

Definition

A **Stein manifold** is a smooth, proper analytic subset of $\mathbb{C}^N$, with the induced complex structure.

Example

$\mathbb{C}^N$ itself is trivially a Stein manifold.

Example

A smooth affine variety is a Stein manifold.

Any Stein n-manifold X admits an exhausting, strictly plurisubharmonic function ρ. Its closed sublevels are called *Stein domains*.

ρ is close to a Morse function with singular points of index $\leq n$, hence $\exists$ *handle decomposition* of X with handles of index $\leq n$.

Example

When $X \subset \mathbb{C}^n$, the square of the radial function $\rho : (z_1, \ldots, z_n) \mapsto \sum |z_j|^2$ is exhausting and strictly plurisubharmonic.

Any regular level set is a *contact manifold*.

Definition

A **contact manifold** is a pair (M^{2n+1}, ξ), where:

- M is an oriented $2n+1$-dimensional smooth manifold;
- $\xi = \ker \alpha$ is a hyperplane field, and α is a 1-form that satisfies $\alpha \wedge d\alpha^n > 0$.

When $M = f^{-1}(r)$ for a regular value r, ξ is given by $J(TM) \cap TM$.

Example

Consider $X = \mathbb{C}^n$, $f = \rho$, $r > 0$: r is regular for ρ, and the corresponding contact manifold is the standard contact $2n-1$-sphere, $(S^{2n-1}, \xi_{\mathrm{st}})$.

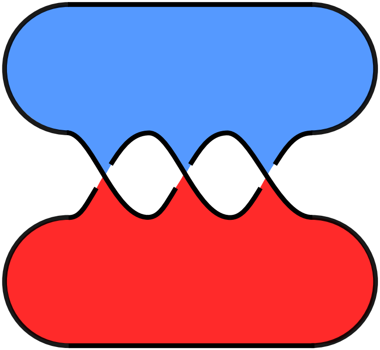
Stein surfaces, (*i.e.* Stein manifolds of complex dimension 2 – real dimension 4) admit a handle decomposition with handles of index 0, 1 and 2.

The 2-handles are attached along *Legendrian knots*.

Definition

A knot L in (M^3, ξ) is **Legendrian** if $TL \subset \xi$.

$L \subset S^3$ topological knot bounds a *Seifert surface*.



The *Seifert genus* of L is $g(L) =$ minimal genus of a Seifert surface.

Let $W = f^{-1}((-\infty, r])$, and suppose f has only one critical point in W, which has index 2.

The attachment of a 4-dimensional 2-handle to B^4 needs:

- A knot: the *attaching circle* L.
- An integer: the *framing* f.

Definition

The **Thurston-Bennequin number** of L is $tb(L) = f + 1$.

The Thurston-Bennequin number of L measures the twisting of the contact structure ξ along L.

Let $W = f^{-1}((-\infty, r])$, and suppose f has only one critical point in W, which has index 2.

$H_2(W;\mathbb{Z}) = \mathbb{Z}$; orienting L gives a generator A.

Definition

The **rotation number** of L is $r(L) = \langle c_1(J), A\rangle$.

The rotation number measures the obstruction of extending the "tangent" trivialisation of $\xi|F$ to a global trivialisation.

Theorem (Bennequin inequality)

$tb(L) + |r(L)| \leq 2g(L) - 1$

Example

For the unknot, $g(\mathcal{O}) = 0$, so $tb(\mathcal{O}) \leq -1$.

Note: there is no Stein structure on $S^2 \times \mathbb{R}^2$ (even though there is a complex structure).

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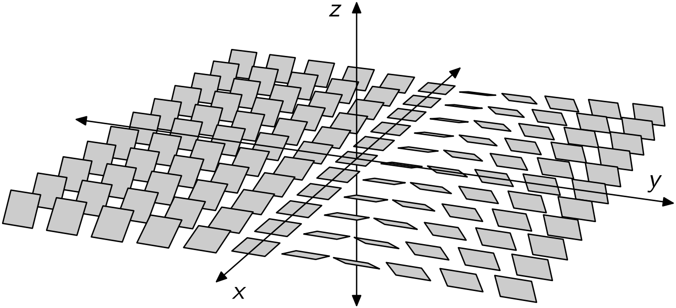
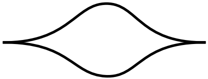
There is no higher-dimensional analogue of Bennequin inequality in higher dimensions: no nontrivial obstructions for the existence of Stein structures (Eliashberg).

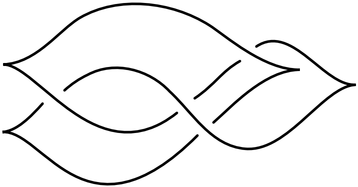
There is a more concrete approach to Legendrian knots.
Removing a point from (S^3, ξ_{st}) yields $(\mathbb{R}^3, \ker(dz - ydx))$.



Source: Wikipedia

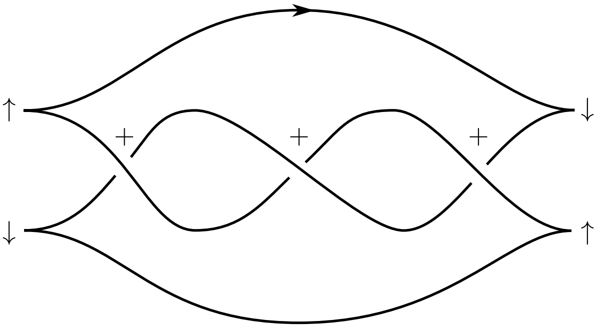
Projecting to the (x, z)-plane gives a nice, visual form for Legendrian knots.





The front projections of a Legendrian unknot and of a right-handed Legendrian trefoil.

One can compute the *classical invariants* *tb* and *r* from the front projection.

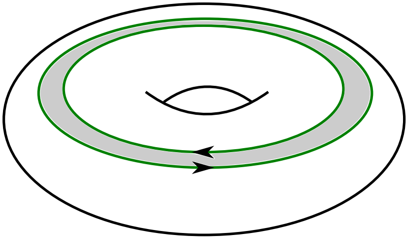


$$tb(L) = wr(L) - c(L)/2$$
$$r(L) = (c^{\downarrow}(L) - c^{\uparrow}(L))/2$$

Juhász defined **sutured Floer homology** $SFH(M,\Gamma)$, that is a finite-dimensional $\mathbb{F}$-vector space associated to a (balanced) sutured manifold (M,Γ).

Example

$L \subset S^3$, N regular neighbourhood of L (*i.e.* a solid torus) and R_+ neighbourhood of a curve on ∂N. $(S^3 \setminus \mathrm{Int}(N), \partial R_+)$ is a sutured manifold.



Legendrian knots have *standard* neighbourhoods.

On $\nu(L)$ there are two parallel, oppositely oriented curves $\gamma_L, -\gamma_L$. Each of this curves *links* $tb(L)$ times with L.
We call S^3_L the sutured manifold $(S^3 \setminus \mathrm{Int}(\nu(L)), \{\gamma_L, -\gamma_L\})$.

Honda-Kazez-Matić defined an invariant $EH(L)$ in $SFH(-S^3_L)$.

Example

For the unknot $\mathcal{O}$ above, $SFH(-S^3_{\mathcal{O}}) = \mathbb{F}_{(0)}$, and $EH(\mathcal{O})$ is the only nonzero element.
For the trefoil L above, $SFH(-S^3_L) = \mathbb{F}_{(1)} \oplus \mathbb{F}_{(0)} \oplus \mathbb{F}_{(-1)}$, and $EH(L)$ is the nonzero element in degree 0.

Ozsváth–Szabó and Rasmussen associate to every (topological) knot L in S^3 a graded $\mathbb{F}[U]$-module $HFK^-(L)$ (multiplication by U lowers grading by 1).

This module is called the **knot Floer homology** of L.

Example

For the unknot $\mathcal{O}$, $HFK^-(\mathcal{O}) = \mathbb{F}[U]_{(0)}$.
For the trefoil $T_{2,3}$, $HFK^-(T_{2,3}) = \mathbb{F}[U]_{(-1)} \oplus \mathbb{F}_{(1)}$.

The knot Floer homology of L is always infinite-dimensional (as a vector space over $\mathbb{F}$).

When L is a Legendrian knot in (S^3, ξ_{st}), there is a class $\mathcal{L}(L)$ in $HFK^-(m(L))$ (Lisca–Ozsváth–Stipsicz–Szabó).

This is an *effective* invariant of Legendrian knots (there is also a combinatorial version).

Example

For $\mathcal{O}$ the unknot above: $HFK^-(m(\mathcal{O})) = \mathbb{F}[U]_{(0)}$, and $\mathcal{L}(L) = 1$ (*i.e.* it generates the free part).
For L the trefoil above: $HFK^-(m(L)) = \mathbb{F}[U]_{(+1)} \oplus \mathbb{F}_{(-1)}$, and $\mathcal{L}(L) = 1$.

There is a related invariant, $\widehat{\mathcal{L}}(L) \in \widehat{HFK}(m(L))$. $\widehat{HFK}(m(L))$ is a finite-dimensional, graded $\mathbb{F}$-vector space.

Example

For the unknot $\mathcal{O}$, $\widehat{HFK}(m(\mathcal{O})) = \mathbb{F}_{(0)}$ and $\widehat{\mathcal{L}}(\mathcal{O}) \neq 0$.

For the trefoil L, $\widehat{HFK}(m(L)) = \mathbb{F}_{(1)} \oplus \mathbb{F}_{(0)} \oplus \mathbb{F}_{(-1)}$ and $\widehat{\mathcal{L}}(L) \neq 0$ has degree 1.

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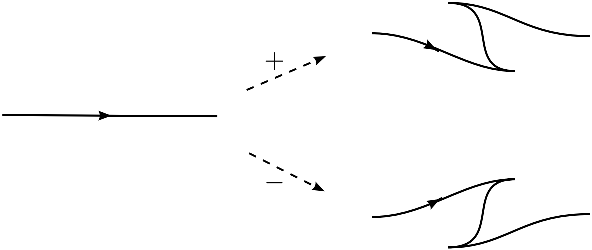
Theorem (Stipsicz–Vértesi)

There is a "natural" map $SFH(-S^3_L) \to \widehat{HFK}(m(L))$ that takes $EH(L)$ to $\widehat{\mathcal{L}}(L)$.

There are two operations on Legendrian knots, called **positive** and **negative stabilisation**.
At the diagram level, one just adds a zig-zag.



If $L^{\pm}$ is a $\pm$ stabilisation of L, then $tb(L^{\pm}) = tb(L) - 1$ and $r(L^{\pm}) = r(L) \mp 1$.

Stabilisations induce maps $\sigma_\pm : SFH(-S^3_L) \to SFH(-S^3_{L^\pm})$, and there are an infinite family of groups $G_n = SFH(-S^3_{L^{(n)}})$ together with maps $\sigma_\pm : G_n \to G_{n+1}$.

Let $G(L) = \varinjlim(G_n, \sigma_-)$.

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Theorem (G.)

- *The group $G(L)$ has an action induced by the map σ_+.*
- $\exists\Psi : G(L) \to HFK(m(L))$*, linear* $\mathbb{F}[U]$*-isomorphism.*
- $\Psi([EH(L)]) = \mathcal{L}(L)$.
- $\mathcal{L}(L)$ *and* $\mathcal{L}(-L)$ *together determine* $EH(L)$.