Knots, polynomials, and categorification

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Knots

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Definition

A knot is the image of a continuous, injective map $S^1 \to S^3 = \mathbb{R}^3 \cup \{\infty\}$.

Example

The map $\iota_h : \theta \mapsto (\cos\theta, \sin\theta, h)$ defines a knot for every $h \in \mathbb{R}$.

We want to consider all these knots to be equivalent, so we define an equivalence relation:

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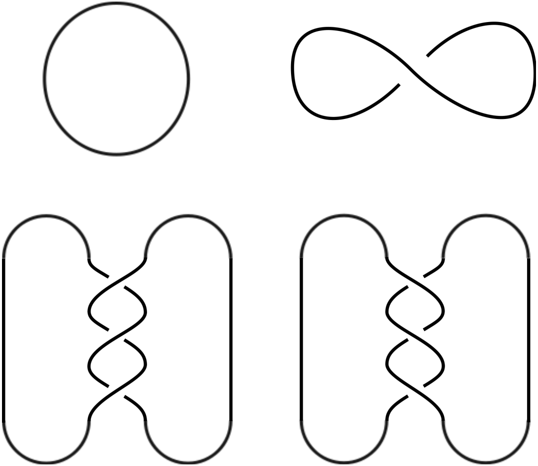
Definition

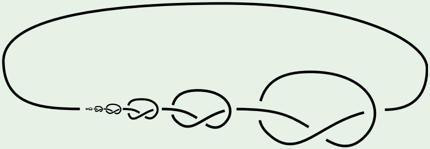
Two knots K_0, K_1 are said to be isotopic if the corresponding maps ι_0, ι_1 are isotopic, *i.e.* there exists a family of continuous, injective maps $\phi_t : S^1 \to S^3$ with $\phi_i = \iota_i$ for $i = 0, 1$.

We can represent a (generic) knot with a projection onto the plane, recording underpasses and overpasses.



Something's wrong

Example



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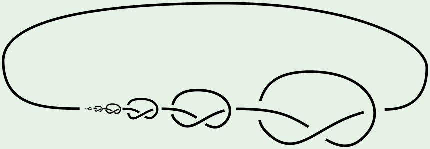


More importantly, every knot is isotopic to any other!

Example

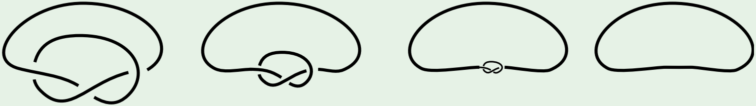


Figure: Pictures of the isotopy taken at times $t = 0,\ 1/2,\ 7/8,\ 1$.

Fixing the definitions

Definition

A knot K is the image of an *embedding* $S^1 \hookrightarrow S^3$.

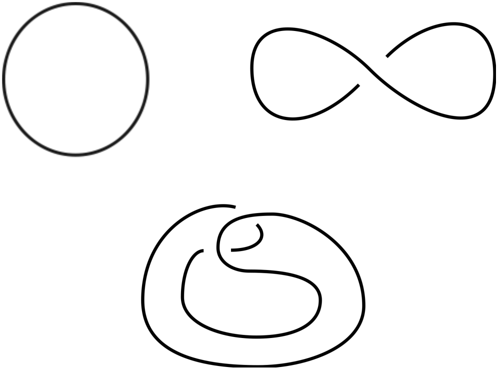
We want to consider a stronger equivalence relation on the space of knots, so as to avoid the squeezing we had before.

Definition

Two knots K^0, K^1 are said to be ambient isotopic if the corresponding maps ι_0, ι_1 are isotopic, *i.e.* there exists a family ϕ_t of self-homeomorphisms of S^3 such that $\phi_0 = \mathrm{id}$ and $\phi_1 \circ \iota_0 = \iota_1$.

Any knot ambient isotopic to $\iota : \theta \mapsto (\cos\theta, \sin\theta, 0)$ is called the unknot.

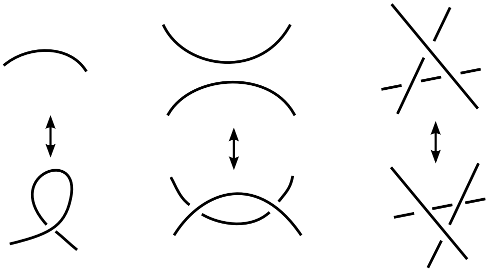
Sometimes different diagrams represent the same knot.



Sometimes different diagrams represent the same knot.

Theorem (Reidemeister, 1926; Alexander-Briggs, 1927)

Two diagrams represent the same knot if and only if one can be obtained from the other through a finite sequence of the following moves:



This is a very theoretical tool!

Knot tables

We can list knots, ordering them by the number of crossings of a minimal projection.

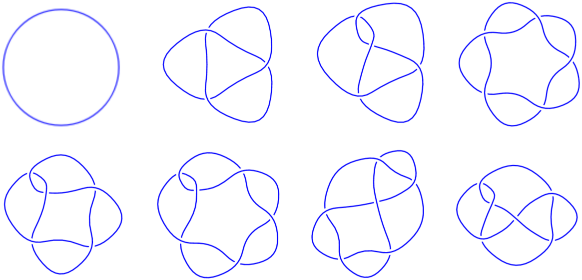


Figure: Pictures taken from KnotInfo

We can list all diagrams (countably many), but we need to make sure we don't make repetitions.

The Perko pair

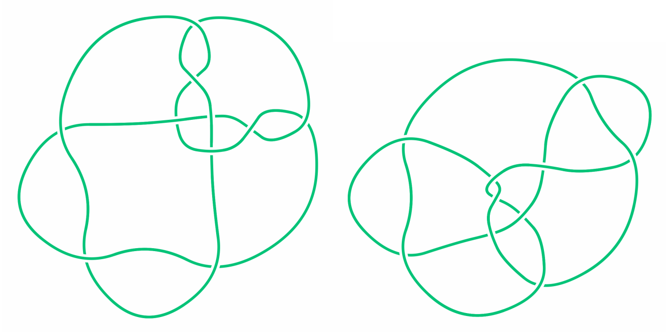


Figure: Pictures taken from wikipedia.

The two knots 10_{161} and 10_{162}.

An knot invariant is a function from the space of knots to some set (naturals, integers, reals, polynomials) or category (groups, vector spaces, manifolds, varieties).

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Example

- The <u>crossing number</u> of a knot K is the minimal number of crossings in a diagram representing K.
- The <u>knot group</u> of K is the fundamental group of the complement $S^3 \setminus K$.

Lemma

Given a knot diagram, one can always switch some crossings to obtain a diagram of the unknot.

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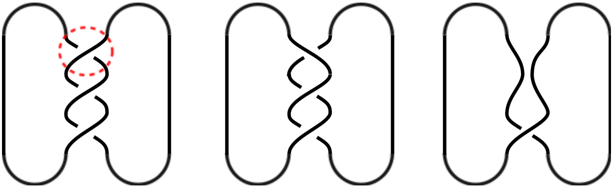
The unknotting number of a knot diagram D is the minimal number of crossings one needs to switch to obtain the unknot.
The unknotting number of a knot K is the minimal knotting number among *all* of the diagrams representing it.



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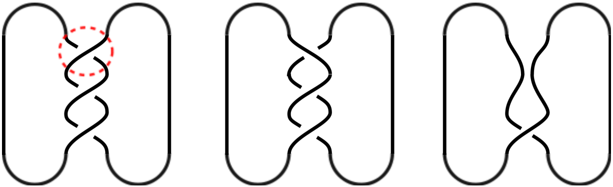

By switching *all* the crossings of a diagram, one obtains (a diagram for) the mirror $m(K)$ of K.

How to define a "computable" knot invariant:

1. Give a recipe to obtain a number or a polynomial from a diagram.
2. Prove that the recipe gives the same number or polynomial if you apply a Reidemeister move.

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Example

A knot K is 3-colourable if one can label the arcs of a diagram for K with red, blue and green, such that

- At each crossings, one sees either all three colours or only one.
- All three colours are used.

Exercise

Prove that this defines an invariant!

Example

The unknot has a diagram with no crossing and one single arc, so every colouring (of *every* diagram representing the unknot) is monochromatic, *i.e.* the unknot is not 3-colourable.

Remark

Every knot has 3 monochromatic 3-colourings.

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Remark

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Proposition (Fox, 1956)

The number of 3-colourings is always a power of 3, and is a knot invariant.

Exercise

Prove the proposition above!

Hint: the three colours can be thought of as elements of $\mathbb{F}_3$...

Proposition

The trefoil knot is not the unknot.

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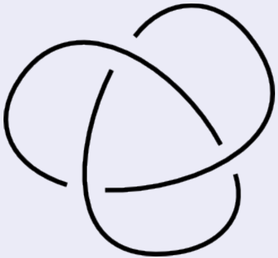
Proof.



□

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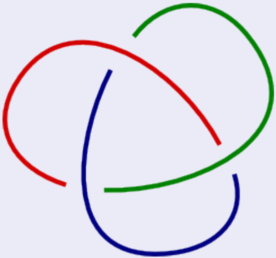

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The Seifert genus

Theorem (Seifert, 1934)

Every knot $K \subset S^3$ bounds an orientable embedded surface.

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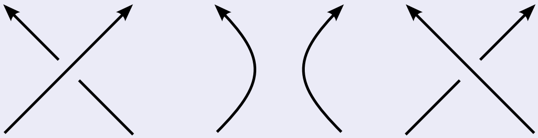
Sketch of proof.

We orient the knot and resolve its crossings by connecting the ends *matching the orientations*.



We obtained a bunch of circles, each of which bounds a disc, and we take the disc together with all the bands. □

Any surface bounding a knot is called a Seifert surface for the knot.
The genus of a surface S with one boundary component is $g(S) := (1 - \chi(S))/2$.
It is always non-negative: $g(S) \geq 0$, with equality if and only if S is a disc.
The genus $g(K)$ of a knot K is the minimal genus of a Seifert surface bounding K.

Example

The unknot has genus 0.
The trefoil has genus 1 (Exercise!)

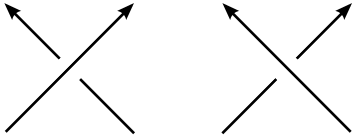
Remark

There are knots for which the minimal genus can't be attained using the algorithm above!

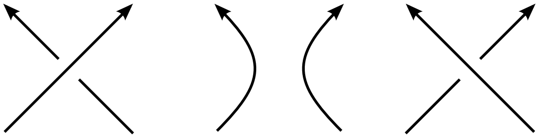
The Alexander polynomial

Let's apply the recipe to cook up invariants in a different way.
Take an *oriented* knot diagram D, and look at a crossing. The crossing can be positive or negative, according to the right-hand rule. We can consider two modifications of D:

- We switch the crossing from negative to positive or vice-versa.



- We resolve the crossing by connecting the ends *matching the orientations*.



We define the Alexander polynomial $\Delta_K(t) \in \mathbb{Z}\left[t, t^{-1}\right]$ "recursively". Given an oriented diagram D for K, we select a crossing, and we let D_+, D_- and D_0 be the diagram where that crossing is positive, negative and resolved respectively.
Then

$$\begin{cases} \Delta_{\bigcirc} = 1 \\ \Delta_{D_+} - \Delta_{D_-} = \left(t^{1/2} - t^{-1/2}\right) \Delta_{D_0} \end{cases}$$

Remark

The definition makes perfect sense for oriented links instead of knots, and in fact we need to consider multiple components to run the algorithm.

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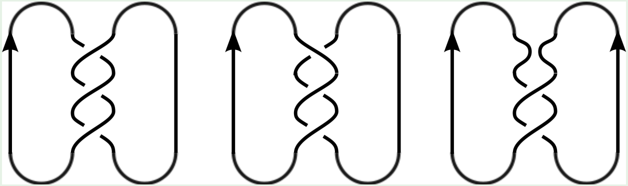
Theorem (Conway, 1969)

Up to multiplication by $\pm t^n$, Δ_K doesn't depend on the diagram D.

Example

We're going to compute the Alexander polynomial of the right-handed trefoil T.

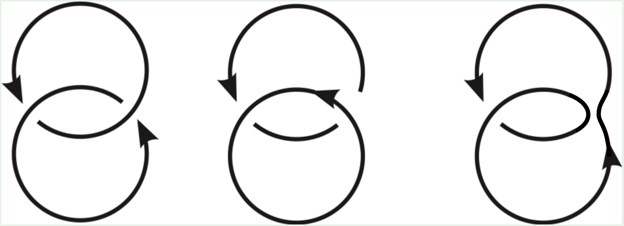


D_+ (representing T), D_- (representing the unknot) and D_0 (representing the (positive) Hopf link).

$$\Delta_T = \Delta_{D_+} = \left(t^{1/2} - t^{-1/2}\right)\Delta_{D_0} + \Delta_{D_-} = \left(t^{1/2} - t^{-1/2}\right)\Delta_{D_0} + 1.$$

Example (Continued)

Let's now compute the Alexander polynomial of the (positive) Hopf link H.



D_+ (now representing H), D_- (representing the unlink with two components) and D_0 (representing the unknot).

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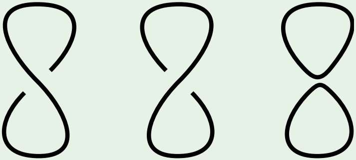
Example (Continued)

Let's now compute the Alexander polynomial of the unlink.



D_+, D_- (both representing the unknot) and D_0 (representing the unlink V).

$$\left(t^{1/2} - t^{-1/2}\right) \Delta_V = \left(t^{1/2} - t^{-1/2}\right) \Delta_{D_0} = \Delta_{D_+} - \Delta_{D_-} = 0.$$

Example (Continued)

Substituting gives:

$$\begin{aligned}\Delta_T &= \left(t^{1/2} - t^{-1/2}\right)\Delta_H + 1 = \\ &= \left(t^{1/2} - t^{-1/2}\right)^2 \Delta_{\bigcirc} - \left(t^{1/2} - t^{-1/2}\right)\Delta_V + 1 = \\ &= t - 1 + t^{-1}.\end{aligned}$$

Remark

The idea of skein relations is that one simplifies the knot (either reducing the number of crossings or the unknotting number or both), and eventually ends up with a bunch of unknots.

Properties of Δ

- If we plug in $t = 1$, we obtain ± 1: $\Delta_K(1) = \pm 1$.

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- The maximal difference of the degrees of the Alexander polynomial is bounded by the genus:

$$\text{max-deg}\,\Delta_K - \text{min-deg}\,\Delta_K \leq 2g(K).$$

From polynomials to vector spaces

Motivation

Consider a finite simplicial complex X (triangulated topological space). We have the Euler characteristic

$$\chi(X) = \sum_{k \geq 0} (-1)^k \#\{k\text{-simplices in } X\},$$

that is an invariant for X (up to homotopy equivalence).

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We can form the vector space $C_k(X)$ generated over $\mathbb{F} = \mathbb{F}_2$ by the k-simplices of X, and define $C_*(X) = \bigoplus_k C_k(X)$.
$C_*(X)$ is *not* an invariant of X up to homeomorphism, but the alternating sum of dimensions is!
Can we make into an invariant?

Motivation (continued)

Define a boundary map $d : C_k(X) \to C_{k-1}(X)$ such that $d^2 = d \circ d = 0$. Let

$$H_k(X) := \frac{\ker\left(d : C_k(X) \to C_{k-1}(X)\right)}{\operatorname{im}\left(d : C_{k+1}(X) \to C_k(X)\right)}$$

$H_*(X)$ is an invariant of X, called simplicial homology.

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Exercise

Prove that

$$\chi(H_*(X)) := \sum_k (-1)^k \dim H_k(X) = \sum_k (-1)^k \dim C_k(X) = \chi(X).$$

We say that "simplicial homology categorifies the Euler characteristic".

Motivation (continued)

To each simplicial map $f : X \to Y$ between simplicial complexes, we associate a map

$$f_* : H_*(X) \to H_*(Y).$$

If f and g are two homotopy equivalent simplicial maps from X to Y, then $f_* = g_*$.
Homology is a functor from the category of triangulable topological spaces to graded vector spaces!

Remark

By making the theory more complicated (from integers to vector spaces) we gain more information.
Moreover, if X has more structure, there are distinguished elements in $H_*(X)$ that χ can't see.

What if we wanted to categorify a polynomial in $\mathbb{Z}\left[t, t^{-1}\right]$?

What if we wanted to categorify a polynomial in $\mathbb{Z}\left[t, t^{-1}\right]$?

For each degree j of the variable t we have an integer a_j (*i.e.* the coefficient of t^j), so for each j we want a graded vector space $V_{*,j}(X)$ so that $\chi(V_{*,j}(X)) = a_j$.
That is, we want to find a *bi*graded vector space

$$V_{*,*}(X) = \bigoplus_{i,j\in\mathbb{Z}} V_{i,j},$$

and we define the (bigraded) Euler characteristic of V:

$$\chi(V) = \sum_{j\in\mathbb{Z}} \left(\sum_{i\in\mathbb{Z}} (-1)^i \dim V_{i,j} \right) t^j \in \mathbb{Z}\left[t, t^{-1}\right].$$

Remark

The i-degree doesn't need to be a $\mathbb{Z}$ grading, but in fact a $\mathbb{Z}/2\mathbb{Z}$ grading is enough.

Guiding principle

diagrams : knots = simplicial complexes : (triangulable) topological spaces

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To cook up an invariant:

1. associate to each knot diagram D a bigraded vector space $V(D)$;
2. define a boundary $\partial : V(D) \to V(D)$;
3. take $\ker \partial / \operatorname{im} \partial$, and hope that it's invariant under Reidemeister moves.

Kauffman states

Consider the regions in which a knot diagram divides the plane.
Declare the "external" region and one adjecent to (*i.e. across an edge from*) it to be forbidden.

Definition

A Kauffman state is any choice of a bijection between the crossings of the diagram and the allowed regions.

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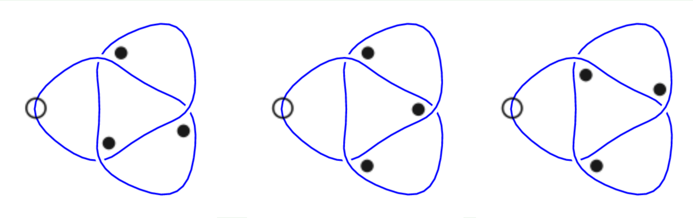
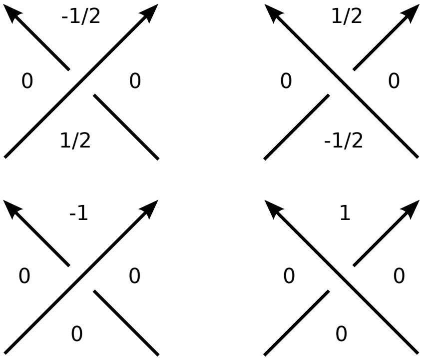
Example



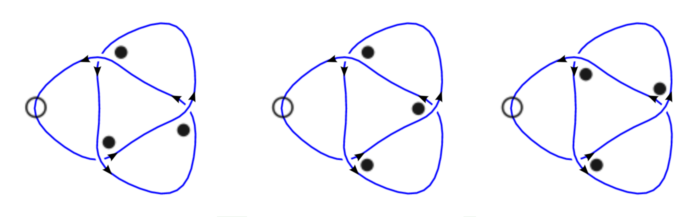
The three states for the standard diagram for the trefoil.

We want to assign two gradings to each state, according to the following rules:



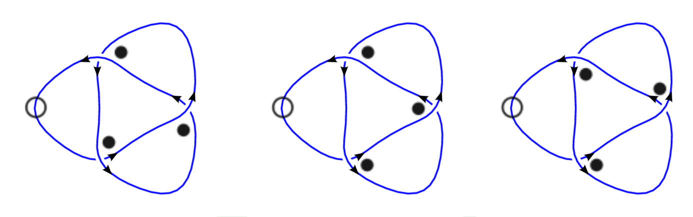
On the top row, the Alexander grading, on the bottom row the Maslov grading.

Example (continued)



The corresponding values of (A, M): $(1, 0)$, $(0, -1)$, $(-1, -2)$.

Example (continued)



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Theorem (Kauffman, 1983)

The weighted count

$$\sum_{i,j} (-1)^i \#\{\textit{states of bidegree } (i,j)\} \cdot t^j$$

is equal to the Alexander polynomial $\Delta_K(t)$

We consider the free vector space $CFK(D)$ generated by the Kauffman states of the diagram D.
$CFK(D)$ is now a bigraded vector space.

Example (continued)

In the example above, we had three generators in bidegrees $(1,0),(0,1)$ and $(-1,0)$.
In this case $CFK(D) = \mathbb{F}_{(1,0)} \oplus \mathbb{F}_{(0,1)} \oplus \mathbb{F}_{(-1,0)}$.

Notice that

$$\chi(CFK(D)) = (-1)^0 t^1 + (-1)^1 t^0 + (-1)^0 t^{-1} = \Delta_T.$$

Remark

$CFK(D)$ is *not* an invariant of the knot K associated to D.

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But we can define an endomorphism $\partial : CFK(D) \to CFK(D)$ that preserves the Alexander grading A, drops the Maslov grading M by 1 and satisfies $\partial^2 = 0$, such that

$$HFK(K) := \frac{\ker \partial : CFK(D) \to CFK(D)}{\operatorname{im} \partial : CFK(D) \to CFK(D)}$$

is an invariant of K.

Theorem (Ozsváth-Szabó, 2002)

$HFK(K)$ is an invariant of K that categorifies the Alexander polynomial, that is

$$\chi(HFK(K)) = \Delta_K(t).$$

Remark

We have that $\dim HFK(K)$ is bounded from below by the sum of the absolute values of the Alexander polynomial.

Example (continued)

In the example above, we had a complex with 3 generators, and we knew that its homology had to be of dimension at least 3.
So in this case $\partial = 0$ and $HFK(T) \simeq CFK(D)$.

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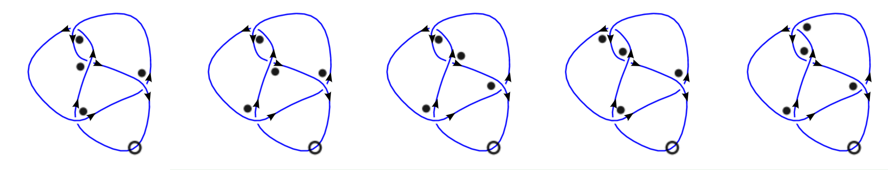
Theorem (Ozsváth-Szabó, 2003)

There's a large class of knots for which knowing the Alexander polynomial (plus the signature of the knot) is equivalent to knowing knot Floer homology. These are called alternating.

Example

Let's consider the figure eight knot $F8$. The states for the 4-crossing diagram are:



With bigradings (A, M): $(-1,-1)$, $(0,0)$, $(1,1)$, $(0,0)$ and $(0,0)$.
In particular, the differential has to be trivial for degree reasons!
It follows that $HFK(F8) = \mathbb{F}_{(1,1)} \oplus \mathbb{F}^3_{(0,0)} \oplus \mathbb{F}_{(-1,-1)}$, and $\Delta_{F8} = -t + 3 - t^{-1}$.

Properties of HFK

- Up to shifts, *HFK* is symmetric:

$$HFK_{M,A}(K) = HFK_{M-2A,-A}(K).$$

Properties of HFK

- Up to shifts, *HFK* is symmetric:

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- *HFK* sees mirrors only through the bigrading:

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- HFK detects the genus:

$$\max\{A \mid HFK_{*,A}(K) \neq 0\} = g(K).$$

By symmetry, one can take $-\min$ instead of max.

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Fill the gap

simplicial complex : continuous map = knot : ???

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Definition

A knot cobordism between K_0 and K_1 is an embedding of a surface F in the cylinder $S^3 \times [0,1]$ such that ∂F is mapped onto $K_0 \times \{0\} \cup K_1 \times \{1\}$.

A knot cobordism is *not* a map, but we can nevertheless compose two cobordisms, provided their sources/targets match, and composition is associative.

Remark

Knots together with knot cobordisms form a category.

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The gap is filled

simplicial complex : continuous map = knot : knot cobordism

Question

Can we associate to a knot cobordisms C between K_0 and K_1 a linear map

$$F_C : HFK(K_0) \to HFK(K_1),$$

so that associativity is respected?

Trivially, yes: if we let $F_C = 0$ for every C, then associativity holds.

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We want one more property: the trivial cobordism

$$K \times [0,1] \subset S^3 \times [0,1]$$

is a two-sided identity with respect to the composition of cobordisms, and it should induce the identity map

$$\mathrm{id} : HFK(K) \to HFK(K)$$

Theorem (Juhász, 2010; Sahamie, 2011)

To every knot cobordism C between K_0 and K_1 one can associate a linear map

$$F_C : HFK(K_0) \to HFK(K_1)$$

in a functorial way.

References

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The Jones polynomial

As before, consider D_+, D_- and D_0, three diagrams differing only at a crossing.
The skein relation defining the Jones polynomial $V_L(q) \in \mathbb{Z}\left[q, q^{-1}\right]$ is:

$$\begin{cases} V_{\bigcirc} = 1 \\ q^{-1} V_{L_+} - q V_{L_-} = \left(q^{1/2} - q^{-1/2}\right) V_{L_0}. \end{cases}$$

Theorem (Jones, 1985)

The skein relation above defines an isotopy invariant of oriented links, with values in $Z\left[q^{1/2}, q^{-1/2}\right]$.

Exercise

Compute the Jones polynomial of the trefoil, as we did before.

Properties of V_K

- The Jones polynomial doesn't see (global) orientation reversals:

$$V_{-K}(q) = V_K(q).$$

- The Jones polynomial *can* see mirroring:

$$V_{m(K)}(q) = V_K\left(q^{-1}\right).$$

- The Jones polynomial sees the number of 3-colourings:

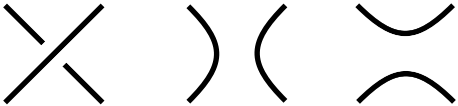
$$c_3(K) = 3\left|V_K\left(e^{i2\pi/6}\right)\right|^2.$$

Conjecture

$V_K = 1$ if and only if K is the unknot.

State-sums

We consider the two (unoriented) resolutions of a diagram:



Call D, D_0 and D_∞ the three diagrams above. The skein relation defining the Kauffman bracket is:

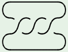
$$\begin{cases} \langle D \rangle = A\langle D_0 \rangle + A^{-1}\langle D_\infty \rangle \\ \langle \coprod^n \bigcirc \rangle = \left(-A^2 - A^{-2}\right)^{n-1} \end{cases}$$

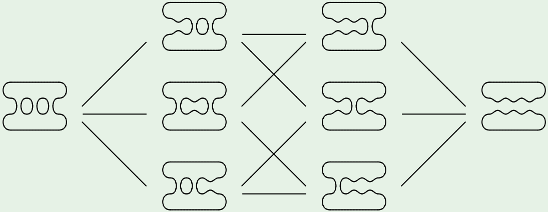
Theorem (Kauffman, 1987)

The Jones polynomial and the Kauffman bracket are related by:

$$V_K(q) = \left((-A)^{-3\,\mathrm{wr}\,D}\langle D \rangle(A)\right)\Big|_{A=q^{-1/4}} \in \mathbb{Z}\left[q, q^{-1}\right].$$

Example

Starting with the right-handed trefoil $T =$  we get the following resolution cube:



Each diagram counts as $(-A^2 - A^{-2})^{\#\,\text{circles}\,-1}$, weighted some power of q (depending on the column it lies in). Let $C = -A^2 - A^{-2}$.

$$V_T(q) = -A^9\left(A^3C^2 + 3AC + 3A^{-1} + A^{-3}C\right) =$$
$$= -q^{-4} + q^{-3} + q^{-1}.$$

Example (continued)

Since $V_T(q)$ is not symmetric (*i.e.* $V_T(q) \neq V_T\left(q^{-1}\right)$), we proved that the right-handed trefoil T and the left-handed trefoil $m(T)$ are *not* ambient isotopic!

We can also check that the formula for the number of 3-colourings holds:

$$V_T\left(e^{2\pi i/6}\right) = -e^{8\pi i/6} + e^{6\pi i/6} + e^{2\pi i/6} = \\ = 2e^{2\pi i/6} - 1 = i\sqrt{3},$$

so that

$$c_3(K) = 3\left|i\sqrt{3}\right|^2 = 9,$$

and this is in fact the case (3 trivial colourings and 6 nontrivial ones).